Stat 206B Lecture 2 Notes

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January 18, 2018

1 Wiener's Theorem, Continued

1.1 Uniform continuity of Brownian motion on the dyadic rationals

Let's finish our proof of the existence of Brownian motion. A good source for this is Freedman's book on Brownian Diffusion.

Theorem 1.1 (Wiener). *On* $(\Omega, \mathcal{F}, P) = ([0,1], \mathcal{B}, Leb)$*, there exists a stochastic process* $(B(t,\omega), t \geq 0, \omega \in \Omega)$ *such that*

1. *For* $0 \leq t_1 < t_2 < \cdots, < t_n$, $B_{t_1}, B_{t_2} - B_{t_1}, \dots, B_{t_n} - B_{t_{n-1}}$ *are independent with mean 0 and variance* $t_1, t_2, -t_1, \dots, t_n - t_{n-1}$ *(this is consistent because* $N(0,s) * N(0,t) \overset{d}{=} N(0, s+t)$*).*

2. $P(\{\omega \in \Omega : t \to B(t,\omega)$ *is a continuous function*$\}) = 1$.

Proof. (cont.) On $(\Omega, \mathcal{F}, P) = ([0,1], \mathcal{B}, \text{Leb})$, we will construct a random path $(B(t,\omega), \omega \in [0,1], t \geq 0)$ such that

1. $(B_t, t \geq 0)$ has the finite dimensional distributions of Brownian motion.

2. For all $\omega \in \Omega$, $t \mapsto B(t,\omega)$ is a continuous function of t.

So far, we have defined $B_t(\omega)$ for t in the dyadic rationals D, and we need only define it for $t \in [0,1]$.

Now we want to show that the function $t \mapsto B_t$ from $D \to \mathbb{R}$ is uniformly continuous with probability 1. We use a technique that is very important in stochastic processes. Let t_n be the greatest multiply of 2^{-n} that is $\leq t$; explicitly, $t = \lfloor t2^n \rfloor / 2^n$. We will show that there exists a sequence of reals $b_n \downarrow 0$ such that

$$P((\exists t \in D \text{ s.t. } |B_t - B_{t_n}| \geq b_n) \text{ i.o.}) = 0.$$

This means that for almost every $\omega \in \Omega$, there exists an $N(\omega)$ such that for all $n \geq N(\omega)$, $|B_t - B_{t_n}| \leq b_n$.

The definition of uniform continuity is $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $|S - t| < \delta \implies |B_s - B_t| < \varepsilon$ for all s, t. Take $\delta \leq 2^{-n}$ so that if $|s - t| \leq \delta$, then every s, t with $|s - t \leq \delta|$ is such that s and t are within the same or adjacent interval of length 2^{-n}. If they are the in the same interval, then they are $\leq b_n$ apart; if s, t are in adjacent intervals, then they are $\leq 2b_n$ apart by the triangle inequality.

By the first Borel-Cantelli lemma, it is sufficient to find $b_n \downarrow 0$ such that

$$\sum_n P(\underbrace{\exists t \in D \text{ s.t.} \, |B_t - B_{t_n}| \geq b_n}_{E_n}) < \infty.$$

Using a union bound,

$$\begin{aligned} P(E_n) &\leq 2^n P(\exists t \in [0, 2^{-n}] \text{ s.t.} \, |B_t| > b_n) \\ &\leq 2^{n+1} P(\exists t \in [0, 2^{-n}] \text{ s.t. } B_t > b_n) \end{aligned}$$

Here, we use a preliminary form of the Brownian motion scaling property that $(B_t, t \geq 0) \xrightarrow{d} =((1/\sqrt{c})B(ct), t \geq 0)$ for all $c > 0$.

$$= 2^{n+1} P(\exists t \in [0, 1] \text{ s.t.} B_t > 2^{n/2} b_n)$$

Note that this is the limit of an union of events, and use the continuity of probability measures.

$$= 2^{n+1} \lim_{k \to \infty} P(\exists t = j.2^k \in [0, 1] \text{ s.t. } B_t > 2^{n/2} b_n)$$

Now we use Levy's maximal inequality applied to centered Gaussians.

$$\begin{aligned} &\leq 2P(B_1 > 2^{n/2} b_n) \\ &= 2^{n+2} \int_{2^{n/2} b_n}^{\infty} \frac{1}{2\pi} e^{-x^2/2} \, dx \end{aligned}$$

We now use a useful bound for the tail of a standard normal distribution, $P(Z > z) \leq \varphi(z)/z$, where φ is the density of Z.

$$= \frac{2^{n+2} \varphi(2^{n/2} b_n)}{2^{n/2} b_n}.$$

So we need only find $b_n \downarrow 0$ such that

$$\sum_{n=1}^{\infty} \frac{2^{n+2} \varphi(2^{n/2} b_n)}{2^{n/2} b_n} < \infty.$$

This is left as an exercise, but you can check that $b_n = 1/n$ works.

We now have on a set of probability 1 that $(B_t, t \in D)$ is uniformly continuous. From analysis, we have the fact that if $f : D \to \mathbb{R}$ is uniformly continuous on a set that is dense in $[0,1]$, then f has a unique continuous extension to a function $g : [0,1] \to \mathbb{R}$. This is done by setting $g(t) := \lim_{d \to t} f(d)$; proof is left as an exercise. So we can define $(B_t, t \in [0,1])$ by an extension of $(B_t, t \in D)$. There is a bad set of probability 0, and there we get $B_t = 0$.

We must check that this B has the correct finite dimensional distributions. The a.s. limit of Gaussians $X_n \sim N(\mu_n, \sigma_n^2)$ (if the limit exists) is a Gaussian random variable with mean $\lim_{n\to\infty} \mu_n$ and variance $\lim_{n\to\infty} \sigma_n^2$, so the property holds. Independence of increments is left as an exercise. □

1.2 Levy's maximal inequality and the significance of Wiener's theorem

We state the lemma used in the above proof here:

Lemma 1.1 (Levy)**.** *Let $S_n = X_1 + \cdots + x_n$ for $X_1, X_2, \ldots$ symmetrically distributed $(X_k \overset{d}{\to} = - X_k)$. Then*

$$P\left(\max_{1 \le k \le n} S_k \ge b\right) \le 2P(X_n \ge b).$$

Why do we study Brownian motion? You might think that it is more useful to study more general continuous random processes, rather than a special case. However, along with Poisson processes, which we will learn about later, we will be able to construct more complicated continuous random processes.

Here are a few calculations we can do before class ends. Let $s < t$. Then

$$\text{Law of } B_t \text{ given } B_s = N(B_s, t-s).$$
$$\text{Law of } B_s \text{ given } B_t = N((s/t)B_s, s(t-s)).$$